Department of Mathematics
Comprehensive Examination
2024 Spring Semester "Choose 2" Classes

Directions: You will answer THREE questions from a total of six questions, posed from two classes. You must answer at least one question from each class.

Time: 2 hours

Math 640: Complex Variables

1. Let $u(x, y) = x^3 - 4x^2 - 3xy^2 + 4y^2$. Prove that u is harmonic and find a harmonic conjugate function $v(x, y)$ such that $u + iv$ is entire.

2. Let S and T be linear fractional transformations.

   (a) Prove that the composition $S \circ T$ is a linear fractional transformation.

   (b) Find a linear fractional transformation which maps $-1 \to 1$, $1 \to \infty$, and $\infty \to -1$.

3. Find the maximum value of $|f(z)| = |z(z - i)(z - 2i)|$ on the closed unit disk

$$D = \{z \in \mathbb{C} \mid |z| \leq 1\}$$

   and find a point in D at which this maximum value is attained.

4. Consider the following maximization problem with associated table showing the beginning and ending tableaux.

$$
\begin{array}{rrrrl}
\max & 10x_1 & + 12x_2 & - 10x_3 & \\
\text{s/t} & 2x_1 & - 6x_2 & + x_3 & = 0 \\
 & x_1 & + x_2 & - x_3 & \leq 10 \\
 & -8x_1 & + 3x_2 & + 4x_3 & \leq 10 \\
 & & & x_1, x_2, x_3 & \geq 0
\end{array}
$$

	x_1	x_2	x_3	x_4	x_5	
a_1	2	−6	1	0	0	0
x_4	1	1	−1	1	0	10
x_5	−8	3	4	0	1	10
	−10	−12	10	0	0	0
x_1	1	0	0	27	5	320
x_2	0	1	0	16	3	190
x_3	0	0	1	42	8	500
	0	0	0	42	6	480

Answer the following questions using techniques of sensitivity analysis. Treat each situation separately, as it refers to the original problem.

(a) What is the range on the coefficient, c_2, of x_2 in the objective function such that the optimal feasible solution of (x_1, x_2, x_3) will not change?

(b) Suppose a new, nonnegative variable, $3x_6$, is added to the original objective function and appears in the first constraint as $-14x_6$ and in the third constraint as $19x_6$. Does the solution change? If so, then find the new solution.

(c) Solve the problem after adding the new constraint

$$x_1 + x_2 - x_3 \leq 8$$

5. Solve the following LP problem.

$$
\begin{array}{rrrrl}
\text{minimize} & 5x_1 & - 25x_2 & + 10x_3 & \\
\text{subject to} & 2x_1 & + 2x_2 & + 2x_3 & \leq 51 \\
 & x_1 & + 3x_2 & + 4x_3 & \leq 100 \\
 & 2x_1 & + x_2 & & \geq 36 \\
 & x_1 & - 4x_2 & + 2x_3 & = 0 \\
 & & & x_j & \geq 0
\end{array}
$$

6. Solve the following transportation problem, using transportation techniques. The numbers along the first column represent the supplies and the numbers along the first row represent the demands.

	10	15	20	25	30	10
50	3	9	5	14	19	13
40	2	3	9	11	18	6
20	13	10	8	3	9	9

Department of Mathematics
Comprehensive Examination
2024 Spring Semester Core Classes

Time: 2.5 hours
Directions:

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620

1. Let G be a group, and K a subgroup of G. Let $g \in G$ and consider the set
$$gKg^{-1} = \{gkg^{-1} : k \in K\}$$
   (a) Prove the set gKg^{-1} is a subgroup of G.
   (b) Prove K is isomorphic to gKg^{-1}.

2. Let G and H be groups, $\varphi : G \to H$ a group homomorphism.
   (a) Prove $\varphi(G) = \{\varphi(g) : g \in G\} \subseteq H$ is a subgroup of H.
   (b) Recall, N is a *normal subgroup* of a group G if $gNg^{-1} = N$ for all $g \in G$. Let J be a normal subgroup of H. Assuming $\varphi(J)^{-1} = \{g \in G : \varphi(g) \in J\}$ is a subgroup of G, prove $\varphi(J)^{-1}$ is a normal subgroup of G.
   (c) Furthermore, assume $\varphi : G \to H$ is a *group isomorphism.* A group G is commutative if $ab = ba$ for all $a, b \in G$. Prove that G is commutative if and only if H is commutative.

3. Let R and S be two nonzero rings with identity and denote the respective identities 1_R and 1_S. Let $\varphi : R \to S$ be a nonzero ring homomorphism.
   (a) Prove that if $\varphi(1_R) \neq 1_S$, then $\varphi(1_R)$ is a zero divisor in S.
   (b) Furthermore, assume $\varphi : R \to S$ is a *ring isomorphism.* Assuming R is an integral domain, prove S is an integral domain.

Math 630

4. Let $\{f_n\}$ be a sequence of continuous real-valued functions on a metric space M that converges uniformly to the function f on M. Prove that f is continuous on M.

5. Show that the interval $(0, 2)$ is not compact in two ways:
   (a) find an open cover with no finite subcover;
   (b) show that the interval $(0, 2)$ is not complete or not totally bounded.

6. (a) Let M be a metric space with metric d and $A \subseteq M$. State what it means for A to be an *open* set.
   (b) Use the definition you have stated to prove that (a, b) is open (where $a, b \in \mathbb{R}$).

7. Let $g(x) = 4\ln(x)$

   (a) Prove that the equation $x = g(x)$ has exactly two positive solutions.

   (b) Consider approximating the larger solution, α, by applying a fixed-point iteration on $g(x)$. Determine an interval $[a, b]$ such that the iteration sequence will converge to α for any initial approximation $x_0 \in [a, b]$. Justify your answer.

8. (a) Explain how to solve the equation $A\mathbf{x} = \mathbf{b}$ using an LU decomposition.

   (b) Write the matrix form you would use to numerically solve the following partial differential equation using backward-difference in time and central-difference in space.

$$\frac{\partial u}{\partial t} - \frac{1}{16}\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1, 0 < t;$$
$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = 2\sin(2\pi x)$$

9. Consider $A = (a_{ij})$ and its splitting $A = D - L - U$ where

$$D = \text{diag}(a_{11}, a_{22}, .., a_{nn}), a_{ii} \neq 0$$

is the diagonal part of A, and

$$-L = \begin{bmatrix} 0 & & & \\ a_{21} & 0 & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & 0 \end{bmatrix}, -U = \begin{bmatrix} 0 & a_{12} & \dots & a_{1n} \\ & \ddots & & \vdots \\ & & 0 & a_{n-1,n} \\ & & & 0 \end{bmatrix}$$

are the lower and upper triangular parts of A respectively. Consider the following linear system:

$$\begin{aligned} 4x_1 - x_2 - x_3 &= 3 \\ -2x_1 + 6x_2 + x_3 &= 9 \\ -x_1 + x_2 + 7x_3 &= -6, \end{aligned} \qquad \mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   (a) The Jacobi method is defined as

$$\mathbf{x}^{(k+1)} = D^{-1}(L+U)\mathbf{x}^{(k)} + D^{-1}\mathbf{b}$$

   or equivalently

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\Big[\sum_{j=1, j\neq i}^{n} \big(-a_{ij}x_j^{(k)}\big) + b_j\Big].$$

   For the linear system above, starting with $\mathbf{x}^{(0)}$,find $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ using the Jacobi method. Will the Jacobi method converge for any intial vector?

(b) The Gauss-Seidel method is defined as

$$\mathbf{x}^{(k+1)} = (D-L)^{-1}U\mathbf{x}^{(k)} + (D-L)^{-1}\mathbf{b}$$

or equivalently

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\Big[\sum_{j>i}^{n}\big(-a_{ij}x_j^{(k)}\big) + \sum_{j<i}\big(-a_{ij}x_j^{(k+1)}\big) + b_j\Big].$$

For the linear system above, starting with $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, find $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ using the Gauss Seidel method. Will the Gauss Seidel method converge for any initial vector?